Intermediate Mathematical Olympiad

Maclaurin paper

supported by

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. Mike is doing a one-hour cycling challenge. He has a computer which predicts how far he will cycle in the rest of the hour based on his average speed so far.
After cycling 1 km in t minutes, he checks the distance the computer predicts he will cycle in the remaining time and it shows d km.
In the next 36 minutes, he cycles 15 km. He checks the computer again and finds it still predicts he will cycle d km in the remaining time.
Find the distance shown on the computer each time he looks.

Solution

Predicted distance is calculated by average speed so far multiplied by the time remaining.

$$d = \frac{60}{t} \times \frac{60 - t}{60}$$
$$d = \frac{16 \times 60}{t + 36} \times \frac{60 - t - 36}{60}$$

These lead to the following equation:

$$\frac{60 - t}{t} = \frac{16(24 - t)}{t + 36}$$

This can be multiplied up and simplified:

$$2160 + 24t - t^2 = 384t - 16t^2$$

This simplifies down to the equation below:

$$(t - 12)^2 = 0.$$

This leads to the only solution being $t = 12$. Substituting this into either of the original equations gives $d = 4$ so the computer shows a predicted distance of 4 km each time.

2. In how many ways can we choose two different integers between -100 and 100 inclusive, so that their sum is greater than their product?

SOLUTION

Let the two integers be m and n and we let $m < n$.

The condition gives the following inequality.

$$m + n > mn$$

This can be rearranged into the useful inequality below because each of the unknowns is now written only once.

$$(m - 1)(n - 1) < 1$$

Given the integer nature of the problem, it can be written in the form below because comparing a product with zero is easier.

$$(m - 1)(n - 1) \leq 0$$

This inequality is true in one of two cases. Firstly, when one bracket is positive and one is negative. Given $m < n$, the first must be negative and the second, positive. Secondly, when one bracket is zero.

In the first case, there are 101 options for m and 99 options for n, so 101×99 options. This gives 9999 cases.

In the second case, there are 200 options when one of them is 1 (99 when $m = 1$ and 101 when $n = 1$).

This means there are $9999 + 200 = 10199$ ways to choose two different numbers between -100 and 100 inclusive so that their sum is greater than their product.

ALTERNATIVE

Consider cases in the original inequality.

If $m < n < 0$, $m + n < 0 < mn$, so this never works.

If $m < n = 0$, we have $m + n = m < 0 = mn$ so this never works.

If $m < 0 < n$, we have $m + n > m \geq mn$, so all these cases work. There are $100 \times 100 = 10000$ of these.

If $m = 0 < n$ we have $m + n = n > 0 = mn$, so this always works. There are 100 of these cases.

If $m = 1 < n$ we have $m + n = 1 + n > n = mn$, so this always works. There are 99 of these cases.

If $1 < m < n$ we have $m + n < n + n = 2n \leq mn$, so this never works.

In total, there are $10000 + 100 + 99 = 10199$ ways.

3. What is the smallest number, n, which is the product of 3 distinct primes where the mean of all its factors is not an integer?

Solution

Let $n = pqr$, where p, q and r are distinct primes.

Its factors are 1, p, q, r, pq, pr, qr and pqr.

The mean of its factors is $\dfrac{1 + p + q + r + pq + pr + qr + pqr}{8}$.

This can be factorised into $\dfrac{(1 + p)(1 + q)(1 + r)}{8}$.

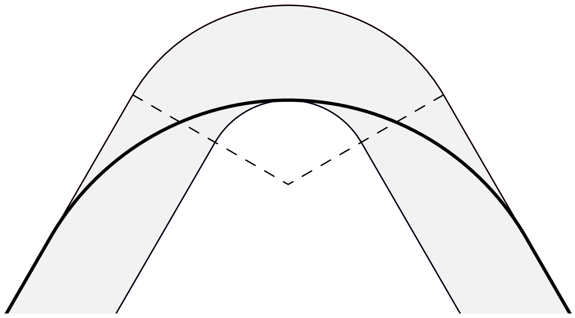
For this to not be an integer the number of factors of 2 in the numerator must be at most 2.

This means at least one bracket must be odd, so one of the primes must be even. However, the only even prime is 2, so one prime must be 2 and the other brackets are both even.

Therefore, neither of the other brackets can have a second factor of 2 and so be a multiple of 4, so the primes cannot be 1 less than a multiple of 4. The smallest two examples of this are 5 and 13.

The smallest value for n is therefore $2 \times 5 \times 13 = 130$.

4. A bend in a road is formed from two concentric arcs with inside radius r and outside radius R, each of a third of a circle with the same centre. The road is then formed of tangents to the arcs.
A cyclist cuts the corner by following an arc of radius x which is tangent to the outside of the road at its ends and tangent to the inside of the road in the middle.

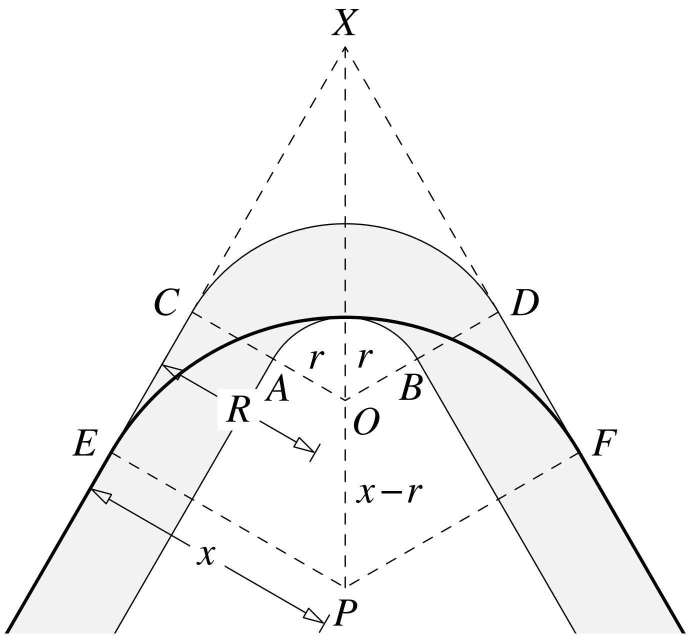


Prove that $r + x = kR$ for some number k to be found.

Solution

Let the centre of the arcs for the curve of the road be O and we label the ends of the arcs.

We extend the tangents to meet at a point X. We draw in the radii from where the cyclist leaves the outside of the curve and label the centre of the arc and the ends of the arc. We join the centres OP and note that this line extended goes through X. The new diagram is shown below.



Note $\angle PEX = \angle OCX = 90°$ because they are tangents. Note POX is the line of symmetry so bisects the angles at O and P. Note $\angle COX = 60°$ because the arcs are a third of a circle and $\angle EPX = 60°$ because PE and OC are both perpendicular to the tangent EX, so they are parallel.

This means triangles PEX and OCX are half of equilateral triangles. Therefore $OX = 2OC = 2R$ and $PX = 2PE = 2x$. Note also that $PO = x - r$ and $PX = 2R + x - r$.

Therefore, $2x = 2R + x - r$ so $x + r = 2R$ and $k = 2$.

Alternative

Drop a perpendicular from O to EP meeting EP at Y and define the point where PO meets the inside of the road as Q, noting that Q is also the centre of the cyclist's path.

Note that $YP = EP - EY = EP - OC = x - R$ and $PO = PQ - OQ = x - r$.

As above, OPY is half of an equilateral triangle, so $PO = 2PY$. Therefore, $x - r = 2(x - R)$ so $x + r = 2R$ and $k = 2$.

Alternative

Extend the tangents at A and B to meet at a point S. Note that, in general, S is not coincident with the outside of the road. Where SA meets EP define the point Z.

As above, OAS is half of an equilateral triangle, so $OS = 2OA = 2r$.

Also, PZS is half of an equilateral triangle, so $PS = 2PZ$. Note that $PS = PO + OS = x - r + 2r = x + r$ and $PZ = PE - EZ = PE - CA = PE - (OC - OA) = x - (R - r) = x - R + r$.

Therefore, $x + r = 2(x - R + r)$ so $x + r = 2R$ and $k = 2$.

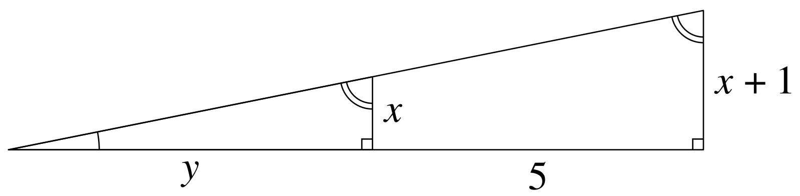
5. Two right-angled triangles are similar. The larger triangle has short sides which are 1 cm and 5 cm longer than the short sides of the smaller triangle. The area of the larger triangle is 8 cm^2 more than the area of the smaller triangle. Find all possible values for the side lengths of the short sides of the smaller triangle.

SOLUTION

Let the side lengths of the smaller triangle be x and y.

There are two cases to consider: the corresponding sides are the same way round or the other way round.

Case 1:



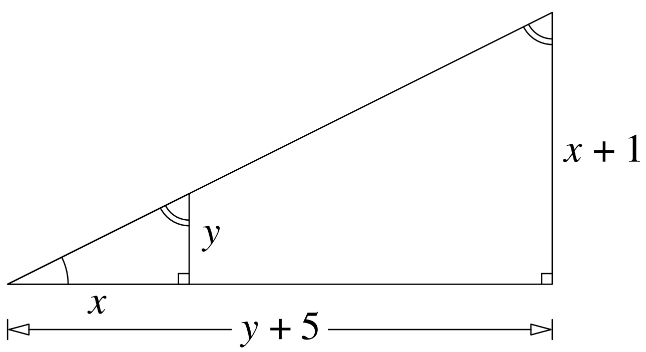
The first case gives the equation for similarity using ratio of sides as $\frac{x}{y} = \frac{x+1}{y+5}$.

The condition on the area gives $\frac{1}{2}xy + 8 = \frac{1}{2}(x+1)(y+5)$.

The first equation becomes $y = 5x$ and this can be substituted into the second equation.

This gives $5x^2 + 16 = 5x^2 + 10x + 5$ which can be solved to give $x = 1.1$ and $y = 5.5$. This is a solution to the problem.

Case 2:



The second case gives the equation for similarity using ratio of sides as $\frac{x}{y} = \frac{y+5}{x+1}$.

The condition on the area still gives $\frac{1}{2}xy + 8 = \frac{1}{2}(x+1)(y+5)$.

The second equation becomes $y + 5x = 11$ and this can be substituted into the first equation.

This gives $x(x+1) = (11-5x)(16-5x)$ which can be simplified to $24x^2 - 136x + 176 = 0$. This can be solved to give $x = 2$ and $y = 1$ or $x = \frac{11}{3}$ and $y = \frac{-22}{3}$. Since solutions cannot be negative, only the first pair is a solution to the problem.

The solutions are $(x, y) = (1.1, 5.5)$ or $(2, 1)$

6. A busy bee buzzes between the cells of a large honeycomb made up of a plane of tessellated hexagons. A flight of length n consists of picking any of the six neighbouring cells and flying to the n^{th} cell in that direction. After consecutive flights of lengths $n = N, N - 1, \ldots, 2, 1$, the bee finds that it has returned to its starting location. For which values of N is this possible?

SOLUTION

If $N = 1$ or $N = 2$, the bee cannot return to its starting position.

If $N = 3$ or $N = 4$, the bee can return by going up and down only. In the first case, it goes 3 up, then 2 and 1 down. In the second case it goes 4 up, then 3 and 2 down, then 1 up.

If $N = 5$ or $N = 6$, the bee can return by forming an equilateral triangle. In the first case, it goes 5 in one direction, 4 and later 1 in another direction and 3 and 2 in the third direction. In the second case, it goes 6 and later 1 in one direction, 5 and later 2 in another direction and 4 and 3 in the third direction.

All further cases for N can be reduced into a case $N - 4$ by going up N, down $N - 1$ and $N - 2$, then up $N - 3$, which returns it to its starting position. This process repeated will bring it down to one of the cases for N being 3, 4, 5 or 6, which can be achieved.

Therefore, for all $N \geq 3$ the bee can return to its starting position.

ALTERNATIVE

As in the first solution, we check small cases and include that for $N = 7$ or $N = 8$, the bee can return by going up, down, down, up to start and then finishing using the results for $N = 3$ and $N = 4$ respectively.

All further cases for N can be reduced into a case $N - 6$ by going in the three directions N and $N - 5$, $N - 1$ and $N - 4$, $N - 2$ and $N - 3$, which returns it to its starting position by forming an equilateral triangle. This process repeated will bring it down to one of the cases for N being 3, 4, 5, 6, 7 or 8 which can be achieved.

Therefore, for all $N \geq 3$ the bee can return to its starting position.

ALTERNATIVE

Define the six hexagons adjacent to the central hexagon as the Ring.

If $N = 1$ or $N = 2$, the bee cannot return to its starting position, but the bee can return from the Ring to the starting position if there are 1 or 2 moves remaining.

Any pair of consecutive moves can result in a move by a single hexagon by reversing the direction for the second move.

Therefore, for any $N \geq 3$, the bee can return to its starting position by the following process. Move to the Ring after two moves. If there are still more than two moves remaining, move to an adjacent cell in the Ring in the next two moves, repeating this until there are one or two moves remaining. Return from the Ring to the starting position in the last one or two moves depending on the parity of N.